Groups, Rings and Representations

Equipo organizador

- J. Miquel Martínez (Universidad de Valencia)
- Noelia Rizo (Universidad de Valencia)
- Lucia Sanus (Universitat de València)

Descripción This session brings together recent advances in the study of finite groups, with a particular focus on character theory, conjugacy classes, and the structure of group rings. The talks cover a wide range of topics, including fields of character values, vanishing elements, class sizes, local-global conjectures, coprime actions, subgroup generation properties, etc.

The session highlights both classical problems and new directions, combining character-theoretic and ring-theoretic methods, illustrating the rich interaction between group structure and representation theory and reflecting the diversity of current research in the area. It aims to foster exchange between researchers working from different perspectives on closely related questions.

Palabras clave: Finite Groups; Characters; Fields of Values; Group Rings; Conjugacy Classes.

Programa

Lunes, 19 de enero

15:30 – 16:00	Antonio Beltrán (Universitat Jaume I) *Cosets of normal subgroups contained in conjugacy classes*
16:00 – 16:30	Carolina Vallejo (Università degli Studi di Firenze) *Coprime actions and blocks*
16:30 – 17:00	Juan Martínez Madrid (Universidad de Valencia) *Commuting probabilty of finite groups*
17:00 – 17:30	Iris Gilabert (Universitat Politècnica de València) *Minimal invariant character degrees of normal subgroups*

Martes, 20 de enero

11:00 – 11:30	Ángel del Río (Universidad de Murcia) *Quadratic groups*
11:30 – 12:00	Sara C. Debón (Universidad de Murcia) *Groups with small fields of values*
12:00 – 12:30	Gabriel A. L. Souza (Universidad de Valencia) *On groups whose character table is almost rational*

15:30 – 16:00	María Dolores Pérez Ramos (Universitat de València) *Two generated subgroups and radical elements*
16:00 – 16:30	Leo Margolis (Universidad Autónoma de Madrid) *The Prime Graph Question for units in integral group rings*
16:30 – 17:00	Rocío Velasco Olalla (CUNEF Universidad) *Some results on Kimmerle's problem*
17:00 – 17:30	Joan Tent (Universidad de Valencia) *Kernels of odd-degree rational irreducible characters*

18:00 – 18:30	Víctor Sotomayor (Universidad de Granada) *Finite groups in which almost all conjugacy class sizes are prime numbers*
18:30 – 19:00	Lucia Sanus (Universitat de València) *On vanishing elements*

Cosets of normal subgroups contained in conjugacy classes

Antonio Beltrán

IMAC, Departamento de Matemáticas, Universitat Jaume I

abeltran@uji.es

Resumen. Let G be a finite group, N a normal subgroup of G, and x an element of G. In 2016, R. Guralnick and G. Navarro proved that if all the elements in the coset Nx are conjugate in G, which is simply equivalent to saying $Nx \subseteq K$ where K is the conjugacy class of x, then N is solvable. We proved that the same conclusion holds when all the elements in Nx lie in the union of a conjugacy class and its inverse class.
However, our investigation shows that N does not necessarily have to be solvable when the coset Nx is contained within the union of any two conjugacy classes of G, say K and D. In this case, the structure of such a non-solvable N is significantly restricted. Using character theory, we establish conditions for a coset Nx to be contained in $K \cup D$ and provide examples that illustrate the limits of our findings. For instance, if K and D have different cardinality then N is solvable.

Groups with small fields of values

Sara C. Debón

Departamento Matemáticas, Universidad de Murcia

sara.cebelland@um.es

Resumen. Several mathematicians have studied the relation between the structural properties of solvable groups and their character values. One of the first results in this direction was made by Gow [1]. He proved that the primes dividing the order of a finite solvable rational group are contained in $\{2, 3, 5\}$. Years later, Hegedüs [2] showed that the Sylow 5-subgroup of a finite solvable rational group is normal and elementary abelian. Naturally, those results have influenced many papers on the topic of groups having relatively small fields of values, such as the quadratic or cut groups, which can be considered as direct generalizations of the rational groups.
In this talk I will discuss some recent advances on the study of these groups. This is joint work with J. Tent.

Referencias

[1] R. Gow, *Groups whose characters are rational-valued*, J. Algebra **40** (1976), 280–299.

[2] P. Hegedüs, *Structure of rational solvable groups*, Proc. London Math. Soc. (3) **90** (2005), 439–471.

Agradecimientos. Partially supported by Agencia Estatal de Investigación of Spain, project PID2020-113206GB-I00 and by Fundación Séneca (22004/PI/22)

Quadratic groups

Ángel del Río

Departamento Matemáticas, Universidad de Murcia

adelrio@um.es

Resumen. We will present some recent developments on finite groups whose ordinary irreducible characters takes values on quadratic extensions of the rationals. This property is connected to the following invariants of a group G: The rationality of G is the following subset of the group $\mathcal{U}_n$ of units of $\mathbb{Z}/n\mathbb{Z}$

$$\mathcal{R}_G = \{r \in \mathcal{U}_n : g \text{ is conjugate to } g^r \text{ in } G \text{ for every } g \in G\}$$

and the semi-rationality of G is

$$\mathcal{S}_G = \{r \in \mathcal{U}_n : \text{ for every } g \in G, \text{ every generator of } \langle g \rangle \text{ is conjugate to } g \text{ or } g^r\}.$$

The field of values of G is

$$\mathbb{Q}(G) = \mathbb{Q}(\chi(g) : g \in G, \chi \in \mathrm{Irr}(G)).$$

The Galois correspondence connects the rationality of G with its field of values and the semi-rationality is connected with the following families of subfields of $\mathbb{Q}(G)$:

$$\{\mathbb{Q}(\chi) = \mathbb{Q}(\chi(g) : g \in G) : \chi \in \mathrm{Irr}(G)\}, \quad \{\mathbb{Q}(g) = \mathbb{Q}(\chi(g) : \chi \in \mathrm{Irr}(G)) : g \in G\}.$$

Agradecimientos. Partially supported by Agencia Estatal de Investigación of Spain, project PID2020-113206GB-I00 and by Fundación Séneca (22004/PI/22)

Minimal invariant character degrees of normal subgroups

Iris Gilabert

Institut Universitari de Matemàtica Pura i Aplicada, Universitat Politècnica de València,

igilman@posgrado.upv.es

Resumen. Let G be a finite group. Recent research shows how the degrees of the minimal G-invariant characters of a normal subgroup N of G exert some influence over its structure, in a way similar to how ordinary irreducible characters influence the structure of a group. For example, if p is a prime dividing the degree of every non lineal minimal G-invariant character of N, then N has a normal p-complement, extending the well-known theorem of Thompson when $N = G$. These characters of N are defined as the sum of irreducible characters of N that belong to the same orbit under the action of G on the set of irreducible characters of N, and they form a basis for the $\mathbb{C}$-vector space of G-invariant class functions of N. In this talk, we present some progress obtained in this current line of research in finite group theory. Joint work with M. J. Felipe and L. Sanus.

Agradecimientos. Partially supported by CIACIF/2023/389 funded the Generalitat Valenciana and the Spanish Ministerio de Ciencia e Innovación (PID2022-137612NB-I00 funded by MCIN/AEI/10.13039/501100011033 and 'ERDF A way of making Europe')

The Prime Graph Question for units in integral group rings

Leo Margolis

Departamento de Matemáticas, Universidad Autónoma de Madrid

leo.margolis@uam.es

Resumen. The unit group of the integral group ring $\mathbb{Z}G$ of a finite group G is a long studied object, but nevertheless many fundamental questions remain open. Concerning the elements of finite order, one could wonder if the existence of a unit of augmentation 1 which has order n implies that also G contains an element of order n. While this is known since the 1960's in case n is a prime power, it remains open even for the case when n is the product of two different primes and is known in this case as the *Prime Graph Question*. While this is a weak variation of many other open questions concerning the unit group of $\mathbb{Z}G$, it has the advantage of admitting a reduction to finite simple groups - thus giving hope a positive solution might be possible.

I will explain a method developed to study the question based on character theory and modular representation theory. It is especially powerful, when a Sylow p-subgroup of G is cyclic, where p is a prime dividing n, though seems useless when p equals 2. Recently we showed how it still can be applied in this situation by considering reductions modulo 4 instead of modulo 2, in some situations when the Sylow 2-subgroup is a Klein four group or dihedral of order 8.

This is joint work with Florian Eisele.

Commuting probabilty of finite groups

Juan Martínez Madrid

Departamento de Matemáticas, Universidad de Valencia

juan.martinez-madrid@uv.es

Resumen. A classical result of W. H. Gustafson asserts that if the probability that two randomly chosen elements of a finite group G commute exceeds $5/8$, then G is abelian. In this talk we present some results which generalize Gustafson's Theorem.

Agradecimientos. Partially supported by Ministerio de Ciencia e Innovación (PID2022-137612NB-I00 funded by MCIN/AEI/10.13039/501100011033 and 'ERDF A way of making Europe') and CIACIF/2021/228 funded by Generalitat Valenciana.

Two generated subgroups and radical elements

María Dolores Pérez-Ramos

Departament de Matemàtiques, Universitat de València

Dolores.Perez@uv.es

Resumen. Roughly speaking, the question how properties of certain subgroups generated by few elements influence the structure of a group is in the core of classical and wide areas of the theory of groups. For instance, the study of (finitely generated) groups all of whose cyclic subgroups are finite gave rise to the classical Burnside problem. Also Thompson's theorem about the solubility of a finite group whose two-generated subgroups are soluble is a remarkable example of how investigation of global properties of groups relies on investigation of their two-generated subgroups. The so-called *one-and-a-half generation theorem* by Guralnick and Kantor plays an important role in this research. It states in particular that no finite simple group has an element that generates proper subgroups with every other element. In the context of products of groups, the study of finite $\mathcal{C}$-connected products for a class of groups $\mathcal{C}$ (i.e. finite factorized groups $G = AB$ such that $\langle a, b\rangle \in \mathcal{C}$ for all $a \in A \leq G$ and all $b \in B \leq G$) provides a more general setting for local-global questions related to two-generated subgroups, by considering the special case when $G = AB = A = B$. Among other relevant results, we highlight an extension of Thompson's theorem from this perspective, obtained in previous research. We report now on new progress in this direction, which involve particularly the class of π-closed subgroups, for a set of primes π.

Collaboration with P. Hauck (U. Tübingen, Germany), L. Kazarin (U. Yaroslavl, Russia), and A. Martínez-Pastor (U. Politècnica de València, Spain).

On vanishing elements

Lucia Sanus

Departament de Matemàtiques, Universitat de València

Lucia.Sanus@uv.es

Resumen. Let G be a finite group, and let $\mathrm{Irr}(G)$ denote the set of the irreducible complex characters of G. An element $g \in G$ is called a *vanishing element* of G if there exists $\chi \in \mathrm{Irr}(G)$ such that $\chi(g) = 0$ (i.e., g is a zero of χ) and, in this case, the conjugacy class g^G of g in G is called a *vanishing conjugacy class*. In this paper we consider several problems concerning vanishing elements and vanishing conjugacy classes; in particular, we consider the problem of determining the least number of conjugacy classes of a finite group G such that every non-linear $\chi \in \mathrm{Irr}(G)$ vanishes on one of them. We also consider the related problem of determining the minimum number of non-linear irreducible characters of a group such that two of them have a common zero.

Joint work with M.L. Lewis, L. Morotti, E. Pacifici and H.P. Tong-Viet.

Agradecimientos. Partially supported by Ministerio de Ciencia e Innovación (PID2022-137612NB-I00 funded by MCIN/AEI/10.13039/501100011033 and 'ERDF A way of making Europe').

Finite groups in which almost all conjugacy class sizes are prime numbers

Víctor Sotomayor

Departamento de Álgebra, Universidad de Granada

vsotomayor@ugr.es

Resumen. Due to reasons that are not fully understood, the conjugacy class sizes of a group share several analogies with its irreducible character degrees. Sometimes these analogies lead to interesting problems. The aim of this talk is to survey some results regarding finite groups that possess few class sizes or character degrees that are composite numbers. In particular, we will show recent progress in the framework of class sizes. This is joint work with C. Monetta (Università di Salerno).

On groups whose character table is almost rational

Gabriel A. L. Souza

Departamento de Matemáticas, Universidad de Valencia

gabriel.area@uv.es

Resumen. There are many known results on groups with few rational conjugacy classes. In this talk, we will look at groups on the opposite extreme, where most conjugacy classes are rational. Among other results, we study the structure of groups with at most 5 irrational conjugacy classes. We also relate the number of irrational conjugacy classes to that of irrational irreducible characters.

Agradecimientos. The author is supported by Ministerio de Ciencia e Innovación (grant PREP2022-000021 tied to the project PID2022-137612NB-I00 funded by MCIN/AEI/10.13039/501100011033 and "ERDF A way of making Europe").

Kernels of odd-degree rational irreducible characters

Joan Tent

Departamento de Matemáticas, Universidad de Valencia

joan.tent@uv.es

Resumen. Rational-valued irreducible characters of odd degree, and their analogues for odd primes, are known to reflect deep properties of the structure of a finite group and have therefore been a topic of interest in character theory over the years. In this talk, we present a generalization of a result conjectured by R. Gow and a refined version of J. Thompson's theorem on character degrees obtained by G. Navarro and P. Tiep, by showing that a finite group possesses a rational, odd-degree irreducible character with 2-nilpotent kernel. Our methods are general enough to apply to any prime, and some additional results which might be of independent interest are required for the odd prime case. This is joint work with A. Moretó.

Agradecimientos. Partially supported by Ministerio de Ciencia e Innovación (PID2022-137612NB-I00 funded by MCIN/AEI/10.13039/501100011033 and 'ERDF A way of making Europe').

Coprime actions and blocks

Carolina Vallejo

Dipartimento di Matematica e Informatica Ulisse Dini, Università degli Studi Firenze

carolina.vallejorodriguez@unifi.it

Resumen. In this talk, we introduce and study the problem of characterizing when an A-invariant block of a group G contains a unique A-invariant irreducible character, in the situation where the group A acts coprimely by automorphisms on G.

This is joint work with Z. Feng.

Some results on Kimmerle's problem

Rocío Velasco Olalla

Departamento de Matemáticas, CUNEF Universidad

rocio.velasco@cunef.edu

Resumen. The Zassenhauss Conjecture on units in group rings asked if any unit of finite order in the integral group ring $\mathbb{Z}G$ of a finite group G is conjugate inside the rational group ring of G to an element of form $\pm g$ for some $g \in G$.
While this has been refuted in general, several versions remain open. In particular, Kimmerle asked if the conjugation can be realized in $\mathbb{Q}H$ for some group H containg G.
We will show some results on Kimmerle's problem (in the class of solvable groups).