Point configurations: energy minimization, discrepancy, and related topics

Equipo organizador

- Ujué Etayo (CUNEF Universidad)
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Descripción

Given a compact topological space—which we may endow with significantly more structure—and a natural number N, which set of N points best represents the space? The answer depends crucially on how we interpret the notion of *representation*. This session brings together researchers who address this question from various perspectives: from geometric combinatorics, looking for cubature nodes and t-designs; from the intrinsic geometry of the space, through the study of point discrepancies; and from classical potential theory, via points that minimize certain energies. Our goal is to foster dialogue across these communities, highlighting connections and uncovering unifying principles in the study of optimal point configurations.

Palabras clave: Point configurations; minimal energy; discrepancy; potential theory; equidistribution.

Programa

Jueves, 22 de enero

11:00 – 11:30	Pablo García Arias (Universitat de Barcelona) *Measuring the equidistribution of point processes via the Wasserstein distance*
11:30 – 12:00	Víctor J. Maciá (CUNEF Universidad) *Some Gaussianity criteria for Khinchin families*
12:00 – 12:30	Joaquim Ortega Cerdà (Universitat de Barcelona) *Fluctuations of eigenvalues of random normal matrices*
12:30 – 13:00	Ujué Etayo (CUNEF Universidad) *Condition number of polynomials and minimizers of Bergman and logarithmic energies*
15:30 – 16:00	Damir Ferizović (KU Leuven) *Point constructions on the sphere with small discrepancy*
16:00 – 16:30	Giacomo Gigante (University of Bergamo) *Irregularities of distribution on compact two-point homogeneous spaces*
16:30 – 17:00	Bianca Gariboldi (University of Bergamo) *Marcinkiewicz–Zygmund families and inverse problems*
17:00 – 17:30	Dmitriy Bilyk (University of Minnesota) *Spherical cap discrepancy, sums of distances, and positive definite functions*
18:00 – 18:30	Jordi Marzo (Universitat de Barcelona) *Hyperuniformity and the Sum Rule*
18:30 – 19:00	Bence Borda (University of Sussex) *Optimal transport of point configurations via harmonic analysis*

Viernes, 23 de enero

11:00 – 11:30	Pedro R. López-Gómez (Universidad de Cantabria) *Points on* SO(3) *with low logarithmic energy*
11:30 – 12:00	Open problem session I
12:00 – 12:30	Open problem session II

Measuring the equidistribution of point processes via the Wasserstein distance

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Resumen. Determinantal Point Processes (abbreviated DPP) are a type of random point processes that have gained some interest due to its applications in physics, its frequent appearance in random matrix theory, and its ability to get uniformly distributed points. The classical way of measuring the latter is using the discrepancy. In recent years the Wasserstein distance has also been considered as another interesting option to do so.

The focus of this talk will be the expected value of the Wasserstein distance W_2 between the empirical measure and the background volume form. This can be studied applying an smoothing via the heat equation, which transforms the problem into finding adequate bounds to the variance of linear statistics of the eigenfunctions associated to the Laplace–Beltrami operator. This technic already works in great generality in high dimension $d \geq 3$, while for dimension $d = 2$ one requires to study each point process with more detail. This is expected, as i.i.d. points obtain optimal asymptotic in these higher dimensions.

This talk will explain how this can be applied to the Harmonic ensemble, variations of it in the torus, the Spherical ensemble, and the zero set of Gaussian Analytic Functions on the sphere. This smoothing technic is able to get optimal asymptotics in all of this cases, despite the different behaviour of the variance – illustrating how this method can work well for different point processes.

References

[1] P. García Arias (2024). Equidistribution of points in the Harmonic ensemble for the Wasserstein distance. *Preprint at* arxiv.org/abs/2405.17298.

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Some Gaussianity criteria for Khinchin families

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Resumen. Generating functions lie at the crossroads of combinatorics, complex analysis, and probability. Given a power series $f(z) = \sum_{n\geq 0} a_n z^n$ with non-negative coefficients and positive radius of convergence $R > 0$, we can turn the coefficients into a *Khinchin family* by declaring, for each $t \in [0, R)$, the mass function

$$\mathbb{P}\big(X_t = n\big) = \frac{a_n\, t^n}{f(t)}, \qquad \text{for any } n \geq 1.$$

This simple prescription simultaneously encodes three objects: the combinatorial numbers a_n, the analytic function f, and a one-parameter family of probability laws $(X_t)_{t\in[0,R)}$. This prompts natural questions: *How do analytic properties of f influence the probabilistic behaviour of X_t?* and, conversely, *what can the probabilistic properties of X_t tell us about the coefficients?*

Beyond providing a probabilistic interpretation of combinatorial sequences, the Khinchin family approach also serves as a powerful tool for asymptotic analysis: under appropriate regularity conditions, probabilistic limit theorems for (X_t) can be translated into precise asymptotic formulas for the coefficients a_n. These regularity conditions are captured by the class of strongly Gaussian power series and the class of Hayman functions. Both classes are contained within the broader class of Gaussian power series, so understanding when a power series is Gaussian is highly relevant for the theory of Khinchin families.

This talk will focus primarily on analytic criteria for Gaussianity in Khinchin families. Starting from the general framework that assigns to any power series with non-negative coefficients a family of probability laws $(X_t)_{t\in[0,R)}$, we investigate when the normalized variables converge in distribution to a standard Gaussian. In particular, we present explicit and verifiable analytical conditions, expressed in terms of derivatives of the fulcrum function associated with f, that ensure this convergence. These results allow us to deduce Gaussian behavior using only the behavior of the analytic function along the positive real axis, yielding a flexible toolkit applicable to a wide range of settings—from classical partition functions to canonical products and exponentials of entire functions of finite order.

References

[1] Maciá, V. J.: Some Gaussianity criteria for Khinchin families. *arXiv preprint* arXiv:2501.04375 (2025).

Fluctuations of eigenvalues of random normal matrices

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Resumen. We consider the fluctuations of the number of eigenvalues of random normal matrices depending on a potential Q in a given set A. These eigenvalues are known to form a determinantal point process, and are known to accumulate on a compact set called the droplet under mild conditions on Q. When A is a smooth set strictly inside the droplet, we show that the variance of the number of eigenvalues N_A in A has a limiting behavior given by

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \operatorname{Var} N_A = \frac{1}{2\pi\sqrt{\pi}} \int_{\partial A} \sqrt{\Delta Q(z)} d\mathcal{H}^1(z),$$

where $d\mathcal{H}^1(z)$ denotes the 1-dimensional Hausdorff measure.

Condition number of polynomials and minimizers of Bergman and logarithmic energies

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Resumen. Given a univariate polynomial with complex coefficients, the condition number of the polynomial at one of its roots quantifies how the root changes under slight perturbations of the polynomial's coefficients. This number may range from 1 (the root remains unchanged) to ∞ (the root has multiplicity greater than 1). In this talk, we present connections between condition number of polynomials and pseudopolynomials, and minimizers of Bergman and logarithmic discrete energies.

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Point constructions on the sphere with small discrepancy

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Resumen. In this talk we give a short summary of discrepancy (mainly on the 2-sphere) and give deterministic constructions of sequences with small L^∞ and L^p discrepancy. We will give an exact parametrization (of almost every) boundary of images of spherical caps under the Lambert cylindrical equal area map which then opens the ally to apply Complex Analytic methods to study questions of discrepancy on the 2-sphere. The point counting functional for instance is then a path integral of the Weierstrass Zeta function in case the point set under consideration is a projected lattice as has been investigated in my previous work [1].

References

[1] D. Ferizović (2024). Spherical Cap Discrepancy of Perturbed Lattices Under the Lambert Projection. Discrete Comput. Geom. 71, 1352–1368 (2024).

Irregularities of distribution on compact two-point homogeneous spaces

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Resumen. I will discuss recent results on the discrepancy of a generic point distribution on a compact two-point homogeneous space, with respect to geodetic balls. In particular, we are interested in lower bounds of the L^2 norm with respect to the centers of the balls, and with one (or two) fixed radius.

Agradecimientos. Joint work with D. Bilyk, L. Brandolini, B. Gariboldi, and A. Monguzzi.

Marcinkiewicz–Zygmund families and inverse problems

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Resumen. Starting from the paper [1] by K. Gröchenig, where the author gives some approximation theorems, we deduce some results for inverse problems in Riemannian manifolds.

References

[1] K. Gröchenig (2020). Sampling, Marcinkiewicz-Zygmund inequalities, approximation, and quadrature rules. *J. Approx. Theory*, 257, 105455.

Agradecimientos. Joint work with G. S. Alberti, E. De Vito, and G. Gigante.

Spherical cap discrepancy, sums of distances, and positive definite functions

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Resumen. Spherical cap discrepancy measures the extent of equidistribution of a discrete set of N points of the sphere and, due to the so-called Stolarsky invariance principle, is closely related to the sum of distances between points. We shall present a new proof of Beck's classical result stating that this discrepancy is always at least of the order $N^{-1/2-1/2d}$. This proof is completely elementary in nature and, unlike the other proofs, avoids using Fourier analysis or spherical harmonics/Gegenbauer polynomials. The argument is also flexible enough to provide inequalities for discrepancy in terms of various other geometric quantities, estimates for discrete Riesz energies, almost sharp constants in the discrepancy bounds, as well as new bounds for discrepancy of lines and general sets of Hausdorff dimension between 0 and d (rather than just point sets). Along the way we shall discuss some related topics that come up: positive definite functions, Welch bounds, frame energy, spherical designs, and energy minimization on the sphere.

Agradecimientos. Joint work with Johann Brauchart (TU Graz).

Hyperuniformity and the Sum Rule

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Resumen. Hyperuniform random point processes have been extensively studied in both the physics and mathematics literature [1]. In Euclidean space, translation-invariant hyperuniform random point processes are characterized by reduced fluctuations. Specifically, the variance in the number of points within a domain grows more slowly than the volume of the domain. Interestingly, in the Euclidean setting, hyperuniform systems can also be characterized by a sum rule, providing an alternative definition of hyperuniformity [2]. In this talk, we will explore hyperuniformity and the sum rule across a variety of geometric settings.

References

[1] S. Torquato (2018). Hyperuniform states of matter. *Physics Reports*, 745:1–95, 2018.

[2] Ph. A. Martin, T. Yalcin (1980). The charge fluctuations in classical coulomb systems. *Journal of Statistical Physics*, 22(4):435–463.

Agradecimientos. Joint work with J. Antezana, M. Levi and J. Ortega-Cerdà (UB). Proyecto PID2024-160033NB-I00 financiado por MICIU/AEIMCIN/AEI/10.13039/501100011033 y por FEDER, UE.

Optimal transport of point configurations via harmonic analysis

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Resumen. The theory of optimal transport and in particular the Wasserstein metric provide a natural way to measure how evenly distributed a finite point configuration is. In this talk, we present a flexible framework to estimate the Wasserstein metric using harmonic analysis on compact manifolds. We also discuss applications to both random and deterministic point sets including determinantal point processes and spherical designs.

References

[1] B. Borda (2023). Empirical measures and random walks on compact spaces in the quadratic Wasserstein metric. *Ann. Inst. Henri Poincaré Probab. Stat.*, 59, 2017–2035.

[2] B. Borda, J. Cuenin (2025). Smoothing inequalities for transport metrics in compact spaces. *Preprint.*

[3] B. Borda, P. Grabner, R. Matzke (2024). Riesz energy, L^2 discrepancy, and optimal transport of determinantal point processes on the sphere and the flat torus. *Mathematika*, 70, Paper No. e12245, 34 pp.

Agradecimientos. Joint work with Jean-Claude Cuenin.

Points on SO(3) with low logarithmic energy

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Resumen. In this talk, I will present a new general strategy to generate points with low logarithmic energy on the special orthogonal group SO(3), that is, the space of 3×3 orthogonal matrices with determinant 1. The main idea behind this construction is the use of the structure of SO(3) as fiber bundle over the two-dimensional sphere $\mathbb{S}^2$ with model fiber $\mathrm{SO}(2) \cong \mathbb{S}^1$. This allows us to generate families of well-distributed points on SO(3) from families of points with that property on the sphere. Following this approach, we construct different families of points on SO(3), all of them with low logarithmic energy. In particular, one of these constructions yields the best upper bound known to date for the logarithmic energy on this space.

References

[1] C. Beltrán, F. Carrasco, D. Ferizović, and P.R. López-Gómez (2025). Points on SO(3) with low logarithmic energy. *Preprint available at* https://arxiv.org/abs/2506.13388.

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