High multiplicity and global structure of coexistence states in a predator-prey model with saturation

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Resumen. In this talk, we analyze the existence, multiplicity and global structure of the coexistence states of the following diffusive one-dimensional Holling–Tanner predator-prey model with Neumann boundary conditions

$$\begin{cases} -u'' = \lambda u - a(x)u^2 - b\frac{uv}{1+\gamma u} & \text{in } (0,1), \\ -v'' = \mu v - dv^2 + c(x)\frac{uv}{1+\gamma u} & \text{in } (0,1), \\ u'(0) = u'(1) = 0, \;\; v'(0) = v'(1) = 0, \end{cases} \tag{1}$$

where $a, c \in \mathcal{C}([0,1];\mathbb{R}) \setminus \{0\}$ are non-negative, $b > 0$ and $d > 0$ are positive constants, and λ, μ and $\gamma > 0$ are viewed as real parameters. Here, $\gamma > 0$ represents the saturation rate of the predator population, v, in the presence of abundance of prey u.

The first aim of this talk is to show that (1) can have an arbitrarily large number of coexistence states for sufficiently large $\gamma > 0$. Namely, considering the change of variables $w := \gamma u$ and $\varepsilon := \frac{1}{\gamma}$, system (1) is equivalent to

$$\begin{cases} -w'' = \lambda w - \varepsilon a(x)w^2 - b\frac{wv}{1+w} & \text{in } (0,1), \\ -v'' = \mu v - dv^2 + \varepsilon c(x)\frac{wv}{1+w} & \text{in } (0,1), \\ w'(0) = w'(1) = 0, \;\; v'(0) = v'(1) = 0. \end{cases} \tag{2}$$

Thus, as $\gamma \uparrow +\infty$, i.e., $\varepsilon \downarrow 0$, system (2) becomes the single equation

$$\begin{cases} -w'' = \lambda w - \frac{b\mu}{d}\frac{w}{1+w} & \text{in } (0,1), \\ w'(0) = w'(1) = 0, \end{cases} \tag{3}$$

Since (3) is an autonomous second order problem, phase plane analysis and time-map techniques can be used to prove the following:

if $\mu \in \left(\frac{d}{b}(2\kappa\pi)^2, \frac{d}{b}(2(\kappa+1)\pi)^2\right]$ for some integer $\kappa \geq 1$, then, the set of positive solutions of (3) consists of the constant solution, w_0, plus κ closed nested loops, $\mathfrak{C}_n$, $n \in \{1, ..., \kappa\}$, bifurcating from w_0 at a family of critical values $\lambda_n^\pm = \lambda_n^\pm(\mu)$ such that

$$0 < \lambda_1^- < \lambda_2^- < \cdots < \lambda_\kappa^- < \frac{b\mu}{2d} < \lambda_\kappa^+ < \cdots < \lambda_2^+ < \lambda_1^+ < \frac{b\mu}{d}.$$

The second aim of this talk is to show that the above high multiplicity result is mantained in the perturbed problem (2) for sufficiently small $\varepsilon > 0$, except, at most, at finitely many values of the parameter λ. This is based on the non-degeneration of the coexistence states of (2) for $\varepsilon = 0$, which relies on the structure of the solution set of (3) and the analyticity of the eigenvalues of the associated variational problems.

Palabras clave: Predator-prey systems of Holling–Tanner type; Multiplicity of coexistence states; Bifurcation Theory; Stability and Phase plane analysis.

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